METHOD OF RELATIVE CORRESPONDENCE AND ITS APPLICATION TO PROBLEMS OF HEAT AND MASS TRANSFER

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The proposed method of analyzing physical phenomena is based on the fact that more approximate models can be used if the relative instead of the absolute quantities characterizing a process are determined. The method is exemplified in the solution of a number of thermophysical problems concerned with the effect of injection and suction on heat transfer in a gas stream, the relation between critical heat load and contact angle, etc.

The investigator of any physical process is usually faced with three problems: analysis of the physical nature of the phenomenon, construction of a model of the process, and mathematical description of the model in the simplest and most convenient form.

Sufficient information for solving the first two problems is not always available (for example, turbulent boundary layer, boiling processes, etc.), while an exact physical formulation often makes the solution very complicated (for example, various types of laminar flows, etc.).

These difficulties can be overcome by using a method that I have called the method of relative correspondence.

It is based on the following postulate: in determining the relative quantities characterizing the deviation of the parameters in two physically similar phenomena it is possible to use much more approximate models of the process than in determining the absolute quantities with the same degree of accuracy.

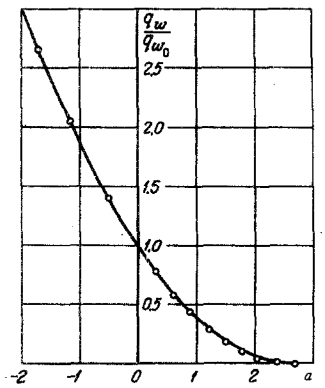

Fig. 1. Effect of injection and suction on the heat transfer of blunt bodies according to the proposed method (solid curve) and the data of [1] (points).

Let us assume that we are interested in some quantity β that depends on a series of parameters

$$\beta = f(x, y, z, \ldots). \tag{1}$$

Its value at certain fixed values of the arguments is known from theory or experiment

$$\beta_0 = f(x_0, y_0, z_0, \ldots). \tag{2}$$

It is asserted that if instead of β, $\bar{\beta} \equiv \beta/\beta_0$ is determined with the same accuracy a more approximate model can be used.

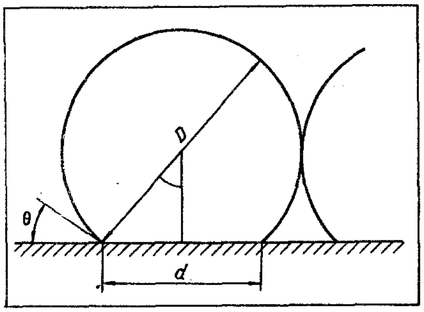

Fig. 2. Diagram illustrating the development of the film boiling regime.

Assume that this model leads to the relation

$$\beta^* = \varphi(x, y, z, \ldots), \tag{3}$$

which differs from the true relation (1).

The series expansion of (1) and (3) in the neighborhood of $(x_0, y_0, z_0, \ldots)$ gives

$$\left.\begin{aligned} \bar{\beta} &= 1 + \frac{1}{f_0}\left(\frac{\partial f}{\partial x}\right)_0 \Delta x + \frac{1}{f_0}\left(\frac{\partial f}{\partial y}\right)_0 \Delta y + \\ &\quad \ldots + \frac{1}{2f_0}\left(\frac{\partial^2 f}{\partial x^2}\right)_0 \Delta x^2 + \ldots, \\ \bar{\beta}^* &= 1 + \frac{1}{\varphi_0}\left(\frac{\partial \varphi}{\partial x}\right)_0 \Delta x + \frac{1}{\varphi_0}\left(\frac{\partial \varphi}{\partial y}\right)_0 \Delta y + \\ &\quad \ldots + \frac{1}{2\varphi_0}\left(\frac{\partial^2 \varphi}{\partial x^2}\right)_0 \Delta x^2 + \ldots. \end{aligned}\right\} \tag{4}$$

It holds trivially for both absolute and relative quantities that the results will be the more accurate, the closer the model to reality and the less x, y, ... differ from x_0, y_0, ...

We note a certain advantage of relative over absolute quantities: for arbitrary Δx, Δy, ... an exact result will be obtained not only when the model and the actual process completely correspond but also when the investigated quantities are proportional, which immediately suggests a greater possibility of arbitrariness in constructing the model.

The above argument, of course, cannot be considered exhaustive, so that we will regard the starting proposition as a postulate whose validity must be demonstrated by its successful application to specific problems.

If it is valid, in analyzing β, the characteristic of a new phenomenon, we take β_0, already known from theory or experiment and fairly close to the studied

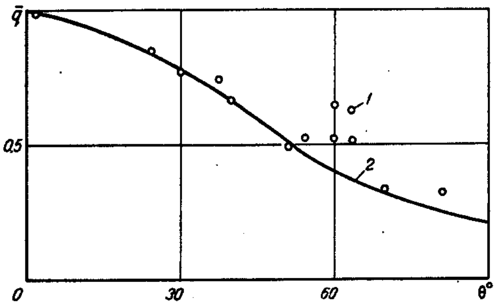

Fig. 3. Critical heat load as a function of contact angle according to the proposed method and the experimental data of [5]: 1) experiment [5]; 2) Eq. (8).

quantity, and create a model which, generally speaking, may be very approximate. Applying it to the old and new variants, we find $\bar{\beta}^*$ and then $\beta \approx \beta_0\, \bar{\beta}^*$. Thus, the proposed method may also be regarded as a method of extrapolating existing data.

I have used this method frequently and, generally, with positive results, with the range of variation of Δx, Δy, ..., within which sufficient accuracy is ensured, often proving very great.

A number of examples illustrating various aspects of the method and with a certain independent interest follow.

§1. Effect of injection and suction on heat transfer in the neighborhood of the forward stagnation point of a blunt body in an incompressible fluid flow. This problem has an exact solution (see, for example, [1]). The corresponding nonlinear differential equation is solved by a rather laborious method using a computer. The final result is given in the form of tables or graphs, which is not very convenient for analytical use. This difficulty may be overcome by means of the proposed method.

Consider a fairly small portion of the flow including the plane or axis of symmetry (x); this portion of the flow impinges on the body through whose surface the liquid is supplied or removed. The change in velocity over this portion of the flow up to the wall itself is assumed to correspond to potential flow, which, generally speaking, is incorrect. Then for the injection or suction of a homogeneous fluid the energy equation and boundary conditions take the form [2]

$$\frac{d^2\bar{T}}{d\bar{x}^2} - \mathrm{Pe}\,\bar{u}\,\frac{d\bar{T}}{d\bar{x}} = 0 \qquad (5)$$

at

$$\bar{x} = 0 \quad \bar{T} = 0, \quad \bar{x} = -\infty \quad \bar{T} = 1,$$

where

$$\bar{x} \equiv \frac{x}{r}, \quad \bar{T} \equiv \frac{T - T_w}{T_\infty - T_w},$$

$$\bar{u} \equiv \frac{u}{u_\infty}, \quad \mathrm{Pe} \equiv \frac{u_\infty r}{\varkappa}$$

(standard notation).

Injection and suction change the boundary conditions for $\bar{u}$, and affect the character of the potential flow. The latter can be accounted for easily, although it was shown [2] that at sufficiently large Pe and small $b \equiv \bar{u}_w/(d\bar{u}/d\bar{x})_{w_0}$ it can be neglected (the subscript 0 corresponds to an impermeable wall).

Solving Eq. (5) for the case of a porous and an impermeable surface, with a sufficient degree of accuracy we obtain [2]

$$\frac{q_w}{q_{w_0}} = \left\{\exp\left(\frac{a^2}{2}\right)[1 + \Phi(a)]\right\}^{-1}, \qquad (6)$$

where $a \equiv -\bar{\rho}\,\bar{u}_w\,(\mathrm{Pe}/(-d\bar{u}/d\bar{x})_{w_0})^{1/2}$ is the injection parameter, and Φ is a tabulated error function.

In order to compare (6) with exact solution we can use the data of [1] for Pr = 0.7 (Fig. 1).

The agreement is excellent over a very broad range of variation of the injection parameter (including both positive and negative values).

It is interesting to note that an equally simple expression is obtained for the case of injection of a different fluid forming a sharp interface with the main stream [2].

This is an example of successful application of the method of relative correspondence for the purpose of overcoming mathematical difficulties.

The following examples relate to cases where the physical nature of the phenomenon is not sufficiently clear.

§2. Dependence of critical heat flux on contact angle. The transition from nucleate to film boiling is a very complex phenomenon, as may be judged from motion-picture records of the process [3].

We will use a simplified model making the following very approximate assumptions:

1) the vapor bubbles formed at the heat transfer surface are spherical, uniform in size, and located at the corners of a square net;

2) the transition from nucleate to film boiling results from the merging of neighboring bubbles and begins at the moment they touch;

3) the effect of the contact angle θ (see Fig. 2) on the heat flux is associated with the different degree of blocking of the surface by the vapor phase, where the heat flux is much less than at points bathed by the liquid.

The macroscopic contact angle θ depends on the nature of the liquid and on the material and state of the surface.

From simple geometric considerations we obtain [4]

$$\bar{q} \equiv \frac{q}{q_0} = \frac{q_f}{q_{f_0}}\left(1 - \frac{\pi}{4}\sin^2\theta\right) \times$$

$$\times \left[1 + \frac{q_s}{q_f}\,\frac{\pi}{4}\sin^2\theta\left(1 - \frac{\pi}{4}\sin^2\theta\right)\right]^{-1}, \qquad (7)$$

where the subscripts f, s, and 0 relate to the liquid, the vapor, and $\theta = 0$, respectively.

At low pressures $q_s \ll q_f$. Moreover, we assume that the heat flow to the liquid phase does not depend on the contact angle. Then (7) simplifies to

$$\bar{q} = 1 - \frac{\pi}{4}\sin^2\theta. \tag{8}$$

Our relations are compared with the experimental data of [5] for water at p = 1 atm abs and $q_0 = 180 \cdot 10^4$ W/m^2.

Figure 3 shows the results, which allowing for the scatter of the experimental data, agree satisfactorily.

Equation (7) predicts a decrease in the influence of the contact angle on critical heat flux with increase in pressure; however, no experimental data are available for verifying this trend.

§3. Turbulent heat and mass transfer on a plate with suction (injection). We consider a porous plate with a turbulent boundary layer through whose surface a gas, generally different from the main-stream gas, is sucked or blown. The resulting mixture is assumed binary. Only concentration diffusion is taken into account, and Pr = Le = 1 is assumed over the entire thickness of the layer.

In this case the profiles of velocity, stagnation enthalpy, and weight fraction of the component introduced are similar, and, moreover, $\mathrm{Nu}/\mathrm{Nu}_0 = C_f/C_{f_0}$, where the subscript 0 refers to an impermeable surface [6].

Our initial, very approximate assumptions concerning the nature of the distribution of shear stress and mixing length over the thickness of the layer are taken from [7]. Strictly speaking, they are valid only near the surface

$$\tau_w = \rho l^2 \left(\frac{du}{dy}\right)^2 - (\rho v)_w u, \quad l = \varkappa y. \tag{9}$$

Moreover, we assume that the thickness of the laminar sublayer is given by

$$u_1/u^* = \alpha_0, \tag{10}$$

where $u^* \equiv ((\tau/\rho)_w)^{1/2}$; the subscript 1 refers to the boundary of the sublayer, and $\alpha \equiv \rho u_1 \delta_1/\mu$.

Relation (6) is selected so that, as $(\rho v)_w \to 0$, $\alpha \to \alpha_0$, and α shows a qualitatively correct dependence on the mode of coolant supply, i. e., decreases for injection and increases for suction.

With these assumptions we determine the velocity profile

$$\ln\left(\frac{y\rho_w u^*}{\mu_w}\right) = \varkappa F + \ln\left\{\frac{\rho_w u^*}{(\rho v)_w}\ln\left[1 + \frac{\alpha_0 (\rho v)_w}{\rho_w u^*}\right]\right\}, \tag{11}$$

where

$$F \equiv \int_{u_1}^{u} \left(\frac{\rho}{\rho_w}\right)^{\frac{1}{2}} \left[u^{*2} + \frac{(\rho v)_w u}{\rho_w}\right]^{-\frac{1}{2}} du.$$

Using the momentum equation and assuming that for a plate

$$\int_0^x c_f dx \simeq \frac{c_f x}{n},$$

where $n \cong 0.8$, we obtain after certain transformations the following expression for the relative coefficient of friction

$$\frac{c_f}{c_{f_0}} + 2n\frac{\bar{w}}{c_{f_0}} = \frac{\sqrt{1 - z_w\left(1 - \frac{1}{\bar{m}}\right)}}{1 - z_w(1 - \bar{c}_p)} \times \frac{\mu_w}{\mu_{w_0}}\frac{I_2}{I_{20}} \times \sqrt{\frac{c_f}{2}\bar{\rho}_w} \times \frac{\ln\left(1 + \alpha_0 \bar{w}\sqrt{\frac{2}{c_f}\frac{1}{\bar{\rho}_w}}\right)}{\alpha_0 \bar{w}}\,\frac{c_{f_0}}{c_f}, \tag{12}$$

where, for brevity, we have used the following notation

$$A^2 \equiv \frac{k-1}{2}\,\frac{M_\infty^2}{\bar{T}_w[1 - z_w(1 - \bar{c}_p)]},$$

$$B \equiv \frac{1}{\bar{T}_w[1 - z_w(1 - \bar{c}_p)]} - 1 + A^2,$$

$$E \equiv \varkappa\left\{\frac{c_f}{2}\bar{T}_w[1 - z_w(1 - \bar{c}_p)]\right\}^{-\frac{1}{2}},$$

$$I_1(\bar{u}) \equiv E\int_0^{\bar{u}}\left[\frac{1 - z_w(1 - \bar{u})\left(1 - \frac{1}{\bar{m}}\right)}{1 - z_w(1 - \bar{u})(1 - \bar{c}_p)} \times \left(1 + 2\frac{\bar{w}}{c_f}\bar{u}\right) \times (1 + B\bar{u} - A^2\bar{u}^2)\right]^{-\frac{1}{2}},$$

$$I_2 \equiv \int_0^1 \left[\frac{1 - z_w(1 - \bar{u})(1 - \bar{c}_p)}{1 - z_w(1 - \bar{u})\left(1 - \frac{1}{\bar{m}}\right)}\right]^{\frac{3}{2}} \times \frac{\bar{u}(1 - \bar{u})\exp[I_1(\bar{u})]\,d\bar{u}}{\left(1 + 2\bar{u}\frac{\bar{w}}{c_f}\right)^{\frac{1}{2}}(1 + B\bar{u} - A^2\bar{u}^2)^{\frac{3}{2}}},$$

$$\bar{w} \equiv \frac{(\rho v)_w}{(\rho u)_\infty};\quad \bar{m} \equiv \frac{m_b}{m_a},\quad \bar{c}_p \equiv \frac{c_{pb}}{c_{pa}},\quad k \equiv \left(\frac{c_p}{c_v}\right)_b, \tag{13}$$

where m is the molecular weight, and the subscripts a and b refer to gases supplied through the pores and the free stream, respectively.

The coolant concentration near the wall is determined from

$$z_w = \left(1 + \frac{1}{2}\frac{c_f}{\bar{w}}\right)^{-1}.$$

The viscosity of the mixture can be obtained using [8].

To avoid solving Eq. (8) by the laborious method of successive approximations with numerical integra-

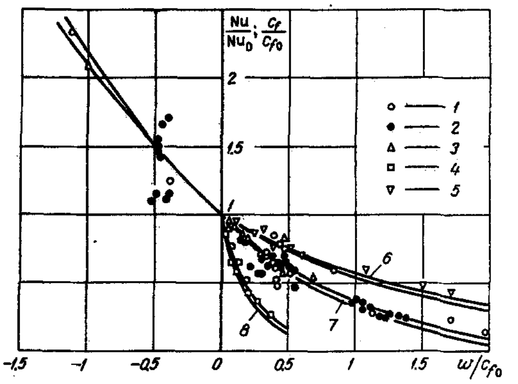

Fig. 4. Heat transfer and friction in a porous plate as a function of the injection parameter according to the proposed method and the experimental data presented in [7] and [9]: 1) air, c_f [7]; 2) air, α [7]; 3) air [9]; 4) helium [9]; 5) freon [9]; 6) freon; 7) air; 8) helium (theory).

tion, it is proposed the use of an approximate method under the following assumptions.

At low injection or suction rates

$$\ln\left(1+\alpha_0\bar{w}\sqrt{\frac{2}{c_f}\frac{1}{\bar{\rho}_w}}\right)\approx\alpha_0\bar{w}\sqrt{\frac{2}{c_f}\frac{1}{\bar{\rho}_w}}.$$

Notice that in this case C_f/C_{f_0} ceases to depend on α_0.

We integrate (13) by parts obtaining a rapidly converging series, such that

$$I_2\approx\frac{1}{E^2}\sqrt{\frac{1-z_w(1-\bar{c}_p)}{1-z_w\left(1-\frac{1}{\bar{m}}\right)}}\Bigg\{1+ +\sqrt{\bar{T}_w\left[1-z_w\left(1-\frac{1}{\bar{m}}\right)\left(1+2\frac{\bar{w}}{c_f}\right)\right]}\times \times\exp I_1(1)\Bigg\},$$

where the unity in the braces can be neglected.

Introducing the new notation

$$K\equiv\frac{E_0}{E}I_1(1)\left\{\frac{c_f}{c_{f_0}}[1-z_w(1-\bar{c}_p)]\right\}^{-\frac{1}{2}}, \qquad (14)$$

$$N\equiv\ln\left\{\frac{[I_1(1)]^2}{1-z_w(1-\bar{c}_p)}\left(\frac{E_0}{E}\right)^2\frac{\mu_{w_0}}{\mu_w}\times \times\frac{\left(1+2n\frac{\bar{w}}{c_f}\right)\sqrt{1-z_w}}{\sqrt{1-z_w\left(1-\frac{1}{\bar{m}}\right)}}\right\}+\frac{E_0}{E}I_{10}(1), \qquad (15)$$

Eq. (12) is reduced to

$$2\ln K+K=N, \qquad (16)$$

which in the range of N values of practical importance can be approximated by the linear expression

$$K_r=m_1N+n_1, \qquad (17)$$

where at $1\le N\le 14.6$, $14.6\le N\le 26.0$, $26.0\le N\le$ ≤ 100 the constants m_1 and n_1, respectively, take the values 0.661, 0.339, 0.879 and −2.84, 0.958, −4.92.

For a more exact determination of K the following approximation can be used

$$K=K_r-(N_r-N)\frac{dK}{dN}= =K_r\left(1-\frac{N_r-N}{K_r+2}\right),$$

where N_r is determined from Eq. (16) after substituting K_r.

Knowing K, we find c_f/c_{f_0} from Eq. (14). The integrals $I_{10}(1)$ and $I(1)$ can easily be tabulated if it is assumed that

$$\frac{1-z_w(1-\bar{u})\left(1-\frac{1}{\bar{m}}\right)}{1-z_w(1-\bar{u})(1-\bar{c}_p)}\approx a+(1-a)\bar{u},$$

where $a\equiv[1-z_w(1-(1/\bar{m}))][1-z_w(1-\bar{C}_p)]^{-1}$, which ensures coincidence of the exact and approximate values of the function at the extremes of the interval $\bar{u}=0$ and $\bar{u}=1$.

These integrals were tabulated [6] for all possible flow regimes and injection parameters.

The results obtained were compared with the experimental data on the suction and injection of air, helium, and freon obtained by Mickley et al. [7] and Pappas and Okuna [9].

Figure 4 shows the experimental and calculated data. For $M_\infty\approx 0$ and $M_\infty=0.7$ the agreement is satisfactory while for $M_\infty\approx 3$ the experimental points lie somewhat higher than the theoretical curves [6].

§4. Law of friction and heat transfer for a turbulent boundary layer. In the previous section we found relative values of the coefficients of friction and heat transfer for injection and suction.

Let us attempt to solve the problem for a more general case using even more approximate assumptions. As before, we compare the regime of interest with a characteristic regime (subscript 0), whose properties are assumed known.

We assume that there is no laminar sublayer and that the function $l=l(y)$ is universal. Using the usual expressions for the shear stress in the layer

$$\tau=\rho\left(l\frac{du}{dy}\right)^2$$

and applying it to the analyzed and characteristic regimes, we obtain

$$\frac{c_f}{c_{f_0}}=\frac{\bar{\rho}}{\bar{\rho}_0}\frac{\bar{\tau}_0}{\bar{\tau}}\left(\frac{\delta_0}{\delta}\right)^2\left[\frac{l}{l_0}\frac{(d\bar{u}/d\bar{y})}{(d\bar{u}/d\bar{y})_0}\right]^2, \qquad (18)$$

where $\bar{\rho}\equiv\rho/\rho_\infty$, $\bar{\tau}\equiv\tau/\tau_w$, $\bar{u}\equiv u/u_\infty$, $\bar{y}\equiv y/\delta$.

If the thicknesses of the dynamic boundary layer δ are the same, then

$$\varphi \equiv \left(\frac{c_f}{c_{f_0}}\right)_\delta = \frac{\bar{\rho}}{\bar{\rho}_0} \frac{\bar{\tau}_0}{\bar{\tau}} \left[\frac{d\bar{u}/d\bar{y}}{(d\bar{u}/d\bar{y})_0}\right]^2.$$

Integration of this equation over the thickness of the boundary layer gives

$$\varphi = \left(\int_0^1 \sqrt{\frac{\bar{\rho}}{\bar{\rho}_0} \frac{\bar{\tau}_0}{\bar{\tau}}}\, d\bar{u}\right)^2. \tag{19}$$

Applying similar reasoning to the thermal boundary layer, we obtain the law of heat transfer

$$\varphi_t \equiv \left(\frac{\mathrm{St}}{\mathrm{St}_0}\right)_{\delta_t} = \left(\int_0^1 \sqrt{\frac{\bar{\rho}}{\bar{\rho}_0} \frac{\bar{c}_p}{\bar{c}_{p_0}} \frac{\bar{q}_0}{\bar{q}}}\, d\bar{u}\right)^2, \tag{20}$$

where we have used the following additional notation: $\bar{C}_p \equiv C_p/C_{p\infty}$, $\bar{q} \equiv q/q_w$, and δ_t is the thickness of the thermal boundary layer.

These laws of friction and heat transfer may be regarded as an extension to the case of an arbitrary characteristic regime of the analogous relations obtained in [10]. The only difference is that in [10] it is arbitrarily assumed that these laws must be satisfied at identical Reynolds numbers with the momentum or energy thickness as the characteristic dimension.

The authors assumed that obtaining such laws is a consequence of the special properties of the turbulent boundary layer as $\mathrm{Re} \to \infty$.

However, since in this case the absolute values of the friction and heat transfer tend to zero, they were obliged to use relative quantities. From our standpoint, it may have been precisely this that ensured the success convincingly demonstrated by the authors for various turbulent boundary layer flow regimes.

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